

Finite topology minimal surfaces in homogeneous three-manifolds

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Abstract

We prove that any complete, embedded minimal surface M with finite topology in a homogeneous three-manifold N has positive injectivity radius. When one relaxes the condition that N be homogeneous to that of being locally homogeneous, then we show that the closure of M has the structure of a minimal lamination of N . As an application of this general result we prove that any complete, embedded minimal surface with finite genus and a countable number of ends is compact when the ambient space is \mathbb{S}^3 equipped with a homogeneous metric of nonnegative scalar curvature.

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1 Introduction.

In this paper we apply the Local Picture Theorem on the Scale of Topology, which is Theorem 1.1 in [5] (see Theorem 1.5 below for the statement of this result in the finite genus setting), to prove that the injectivity radii of certain minimal surfaces in certain Riemannian three-manifolds are never zero.

Theorem 1.1 *Let N be a complete, locally homogeneous three-manifold with positive injectivity radius. Then, every complete, embedded minimal surface of finite topology in N has positive injectivity radius.*

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In the case that the ambient three-manifold N is isometric to $\mathbb{S}^2 \times \mathbb{R}$ with a scaling of its standard product metric, then this result follows from Theorem 15 in [10] where Meeks and Rosenberg also applied Theorem 1.1 in [5] to prove the stronger property that such minimal surfaces have bounded second fundamental form, linear ambient area growth and so they are also proper in $\mathbb{S}^2 \times \mathbb{R}$. For background material on the geometry and classification of homogeneous three-manifolds, see [4].

The next result removes the positive injectivity radius assumption for the ambient space N . The conclusion that we obtain in this setting is also weaker than the one in Theorem 1.1, as follows from the Minimal Lamination Closure Theorem in [10].

Corollary 1.2 *If M is a complete embedded minimal surface of finite topology in a complete, locally homogeneous three-manifold N , then the closure \overline{M} has the structure of a minimal lamination of N . Furthermore:*

1. *Each limit leaf of \overline{M} is stable (more precisely, the two-sided cover of the leaf is stable).*
2. *If N has positive scalar curvature, then M is proper in N .*
3. *If N is simply connected and has nonnegative scalar curvature, then M is proper in N .*
4. *If N is the round three-sphere \mathbb{S}^3 , then M is compact.*

Remark 1.3 Item 1 of Corollary 1.2 still holds without the hypothesis on N to be locally homogeneous. On the other hand, it can be shown that there exists a Riemannian metric of positive scalar curvature on the three-sphere that admits a complete embedded minimal plane whose closure does not admit the structure of a minimal lamination (see e.g., [1]). Hence, our hypothesis that N is locally homogeneous is necessary for items 2, 3 of Corollary 1.2 to hold.

In the sequel, we will denote by $B_M(p, r)$ (resp. $\overline{B}_M(p, r)$) the open (resp. closed) metric ball centered at a point p in a Riemannian manifold M , with radius $r > 0$. In the case M is complete, we will let $I_M: M \rightarrow (0, \infty]$ be the injectivity radius function of M , and given a subdomain $\Omega \subset M$, $I_\Omega = (I_M)|_\Omega$ will stand for the restriction of I_M to Ω . The infimum of I_M is called the *injectivity radius* of M .

We now briefly explain the main tool used in our proofs, namely Theorem 1.1 in [5], in the special case of surfaces of finite genus in a homogeneously regular¹ three-manifold. The paper [5] was devoted to an analysis of the extrinsic geometry of any embedded minimal surface M (not necessarily with finite genus) in small intrinsic balls in a homogeneously regular Riemannian three-manifold, such that the injectivity radius function of M is sufficiently small

¹A Riemannian three-manifold N is *homogeneously regular* if there exists an $\varepsilon > 0$ such that the image by the exponential map of any ε -ball in a tangent space $T_x N$, $x \in N$, is uniformly close to an ε -ball in \mathbb{R}^3 in the C^2 -norm. In particular, N has positive injectivity radius. Note that if N is compact, then N is homogeneously regular.

in terms of the ambient geometry of the balls. We carried out this analysis by blowing-up such an M at a sequence of points with *almost-minimal injectivity radius* (we will define this notion precisely in items 1, 2, 3 of the next theorem), which produces a new sequence of minimal surfaces, a subsequence of which has a natural limit object being either a properly embedded minimal surface in \mathbb{R}^3 , a minimal parking garage structure on \mathbb{R}^3 (see Section 3 of [5] for a discussion of this notion) or possibly, a particular case of a singular minimal lamination of \mathbb{R}^3 with restricted geometry; an important property is that this last possibility can occur only if M fails to have locally finite genus, in the sense given by the following definition.

Definition 1.4 A Riemannian surface M has *locally finite genus* if there exists an $\varepsilon > 0$ such that intrinsic balls in M of radius ε have uniformly bounded genus.

The following result is an adaptation of Theorem 1.1 and Proposition 4.20 in [5] to the case of M having locally finite genus.

Theorem 1.5 (Local Picture on Scale of Topology for Locally Finite Genus) *There exists a smooth decreasing function $\delta: (0, \infty) \rightarrow (0, 1/2)$ with $\lim_{r \rightarrow \infty} r \delta(r) = \infty$ such that the following statements hold. Suppose M is a complete, embedded minimal surface with injectivity radius zero and locally finite genus in a homogeneously regular three-manifold N . Then, there exists a sequence of points $p_n \in M$ (called “points of almost-minimal injectivity radius”) and positive numbers $\varepsilon_n = n I_M(p_n) \rightarrow 0$ such that:*

1. *For all n , the closure M_n of the component of $M \cap B_N(p_n, \varepsilon_n)$ that contains p_n is a compact surface with boundary in $\partial B_N(p_n, \varepsilon_n)$. Furthermore, M_n is contained in the intrinsic open ball $B_M(p_n, \frac{r_n}{2} I_M(p_n))$, where $r_n > 0$ satisfies $r_n \delta(r_n) = n$.*
2. *Let $\lambda_n = 1/I_M(p_n)$. Then, $\lambda_n I_{M_n} \geq 1 - \frac{1}{n}$ on M_n .*
3. *The metric balls $\lambda_n B_N(p_n, \varepsilon_n)$ of radius $n = \lambda_n \varepsilon_n$ converge uniformly as $n \rightarrow \infty$ to \mathbb{R}^3 with its usual metric (so that we identify p_n with $\vec{0}$ for all n).*

Furthermore, exactly one of the following two possibilities occurs.

4. *The surfaces $\lambda_n M_n$ have uniformly bounded Gaussian curvature on compact subsets² of \mathbb{R}^3 and there exists a connected, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$ and $I_{M_\infty}(\vec{0}) = 1$, such that for any $k \in \mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of \mathbb{R}^3 to M_∞ with multiplicity one as $n \rightarrow \infty$.*
5. *After a rotation in \mathbb{R}^3 , the surfaces $\lambda_n M_n$ converge to a minimal parking garage structure on \mathbb{R}^3 , consisting of a foliation \mathcal{L} of \mathbb{R}^3 by horizontal planes, with two oppositely handed columns forming a set $S(\mathcal{L})$ of two vertical straight lines (the set $S(\mathcal{L})$ is the singular set of convergence of $\lambda_n M_n$ to \mathcal{L} , see Definition 1.6 below), and:*

²As $M_n \subset B_N(p_n, \varepsilon_n)$, the convergence $\{\lambda_n B_N(p_n, \varepsilon_n)\}_n \rightarrow \mathbb{R}^3$ explained in item 3 allows us to view the rescaled surface $\lambda_n M_n$ as a subset of \mathbb{R}^3 . The uniformly bounded property for the Gaussian curvature of the induced metric on $M_n \subset N$ rescaled by λ_n on compact subsets of \mathbb{R}^3 now makes sense.

- (5.1) $S(\mathcal{L})$ contains a line l_1 which passes through the closed ball of radius 1 centered at the origin, and another line l_2 at distance one from l_1 .
- (5.2) There exist oriented closed geodesics $\gamma_n \subset \lambda_n M_n$ with lengths converging to 2, which converge to the line segment γ that joins $(l_1 \cup l_2) \cap \{x_3 = 0\}$ and such that the integrals of the unit conormal vector of $\lambda_n M_n$ along γ_n in the induced exponential \mathbb{R}^3 -coordinates of $\lambda_n B_N(p_n, \varepsilon_n)$ converge to a horizontal vector of length 2 orthogonal to γ .

Definition 1.6 If $\{\Sigma_n\}_n$ is a sequence of complete embedded minimal surfaces in a Riemannian three-manifold N , consider the closed set $A \subset N$ of points $p \in N$ such that for every neighborhood U_p of p and every subsequence of $\{\Sigma_{n(k)}\}_k$, the sequence of norms of the second fundamental forms of $\Sigma_{n(k)} \cap U_p$ is not uniformly bounded. By the arguments in Lemma 1.1 of Meeks and Rosenberg [9], after extracting a subsequence, the Σ_n converge on compact subsets of $N - A$ to a minimal lamination \mathcal{L}' of $N - A$ that extends to a minimal lamination \mathcal{L} of $N - \mathcal{S}$, where $\mathcal{S} \subset A$ is the (possibly empty) *singular set* of \mathcal{L} , i.e., \mathcal{S} is the closed subset of N such that \mathcal{L} does not admit a local lamination structure around any point of \mathcal{S} . We will denote by $S(\mathcal{L}) = A - \mathcal{S}$ the *singular set of convergence* of the Σ_n to \mathcal{L} , i.e., those points of N around which \mathcal{L} admits a lamination structure but where the second fundamental forms of the Σ_n still blow-up.

2 Proof of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. Let M be a complete, embedded minimal surface of finite topology in a complete, locally homogeneous three-manifold N with positive injectivity radius. Since M has finite topology, then it has a finite number of ends, all being topologically annuli. If the injectivity radius function I_M of M is bounded away from zero on each of its annular ends, then I_M is globally bounded away from zero. Hence, the theorem will follow provided that we show that I_M is bounded away from zero on each end of M , or equivalently, if each end representative E of M satisfies the following property:

(End) If $f: E = \mathbb{S}^1 \times [0, \infty) \rightarrow N$ is a complete injective minimal immersion and N satisfies the hypotheses of Theorem 1.1, then the injectivity radius function I_E of E is bounded away from zero in the complement of any neighborhood of the boundary ∂E .

Our strategy to prove property (End) will be first prove it in the particular case when N is simply connected (Assertion 2.1 below) and then use this particular case to demonstrate the general case (Assertion 2.2).

Assertion 2.1 *If N is simply connected, then property (End) holds.*

Proof. As N is locally homogeneous, complete, simply connected and has dimension three, then N is homogeneous. In this setting and as shown in [4], N is isometric to either $\mathbb{S}^2 \times \mathbb{R}$ with a scaling of its standard metric or to a three-dimensional metric Lie group, i.e., a Lie group endowed with a left invariant metric. If $N = \mathbb{S}^2 \times \mathbb{R}$ with a scaling of its standard metric, then we refer the reader to [10] that contains the stronger result that the second fundamental form of a complete annular minimal end E is bounded; also see the discussion just after the statement of Theorem 1.1. Therefore, in the sequel we will assume that N is a metric Lie group.

For simplicity, we let $E = f(E)$ denote the embedded minimal annulus. If I_E is not bounded away from zero outside of some neighborhood of ∂E , then there exists an intrinsically divergent sequence of points $p_n \in E$, with $I_E(p_n) < \frac{1}{n}$, $d_E(p_n, p_{n+1}) > n$; such a sequence of points p_n can be chosen to be points of almost-minimal injectivity radius. By Theorem 1.5, after choosing a subsequence, the local picture on the scale of topology of E around the sequence p_n is either a minimal parking garage structure of \mathbb{R}^3 with two oppositely handed columns (that is, item 5 of Theorem 1.5 occurs), or a properly embedded minimal surface M_∞ with genus zero (item 4 of Theorem 1.5 occurs). In this last case, M_∞ is a catenoid or a Riemann minimal example by classification results (see [8] and references therein).

We claim that the sequence of points p_n is diverging in N . Arguing by contradiction, we may assume, after replacing by a subsequence, that the points p_n converge to a point $p \in N$. Using normal coordinates around p and taking a further subsequence we may also assume that the rescaled space in which the local picture of E exists is \mathbb{R}^3 with induced coordinates, and that the flux vector of the rescaled limit object is not zero and parallel to $(0, 0, 1)$; see item 2 of Theorem 1.5 for the definition of this flux vector in the case that the limit object is a parking garage structure on \mathbb{R}^3 , i.e., it is not a catenoid or a Riemann minimal example in which cases the flux vector is just the flux vector for any of the circles on the limit surface. Let V be the right invariant vector field on N determined by $V(p) = (0, 0, 1)$ (with respect to a previously chosen orthonormal basis of $T_p N$ so that under rescaling of the metric, the direction of V at p gives rise to the vertical direction in the limit \mathbb{R}^3). Note that since the metric on N is left invariant, then every right invariant vector field on N is a Killing field. Let V^T denote the tangential part of V on E . Consider the (scalar) flux of V^T across any oriented closed curve Γ in E , defined as

$$\text{Flux}(V^T, \Gamma) = \int_{\Gamma} \langle V, \eta \rangle, \quad (1)$$

where \langle, \rangle stands for the ambient metric on N and η is a unit conormal to E along Γ .

Since the mean curvature of E is zero and V is a Killing field, then the divergence of V^T in E vanishes identically, and an elementary application of the divergence theorem implies that $\text{Flux}(V^T, \Gamma)$ is a homological invariant of Γ . Choose simple closed oriented curves $\Gamma_n \subset E$ near p_n for n large, so that $\text{Flux}(V^T, \Gamma_n) \neq 0$, which can be done since after rescaling, the vertical component of the flux vector of the limit object is not zero. As the homology group $H_1(E)$ is \mathbb{Z} , we deduce that the Γ_n represent one of the two nontrivial generators of $H_1(E)$

and that the Γ_n can be oriented so that $\text{Flux}(V^T, \Gamma_n) = \text{Flux}(V^T, \partial E)$ for all n . But by the injectivity radius zero assumption on E , these curves Γ_n can be chosen to have lengths converging to zero, which is impossible because the Γ_n converge to p and $|V(p)| = 1$ and so $|\text{Flux}(V^T, \Gamma_n)| \leq 2 \text{length}(\Gamma_n)$. This contradiction implies that the blow-up points p_n are divergent in N .

The proof of Assertion 2.1 now breaks up into 3 cases, depending on the ambient space.

Case A: N is the special unitary Lie group $SU(2)$ with a left invariant metric.

Case B: N is the Lie group $\widetilde{\text{PSL}}(2, \mathbb{R})$ (the universal cover of the group of isometries of the hyperbolic plane) with a left invariant metric.

Case C: N is a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some real 2×2 matrix A , equipped with a left invariant metric; see Section 2.1 in [4] for details. This case includes all the noncompact, simply connected, nonunimodular metric Lie groups as well as the Heisenberg group Nil_3 , the solvable group Sol_3 , the universal cover $\widetilde{E}(2)$ of the Euclidean group of isometries of \mathbb{R}^2 and the abelian Lie group \mathbb{R}^3 , each of them equipped with any left invariant metric.

Proof of Case A: This case follows immediately from the fact that $SU(2)$ is compact and the just proved fact that whenever Assertion 2.1 fails, there exists a sequence of points $p_n \in E$ of almost-minimal injectivity radius that diverges in the ambient space.

Proof of Case B: Let $G: E \rightarrow \mathbb{S}^2$ denote the left invariant Gauss map for E ; by this we mean the mapping valued in the unit sphere of the tangent space $T_e N$ to N at the identity element $e \in N$ (or equivalently, the Lie algebra \mathfrak{g} of the metric Lie group N), which assigns to each point $p \in E$ the left translation $G(p) \in T_e N$ of the unit normal vector field ν_p to E at p , i.e., $(l_p)_* G(p) = \nu_p$, where $l_p: N \rightarrow N$ denotes the left translation by p and $(l_p)_*$ is its differential. As explained in the second paragraph of the proof of the assertion, after rescaling E on the scale of topology around a sequence of points $p_n \in E$ of almost-minimal injectivity radius, we find a limit object which is a catenoid, a Riemann minimal example or a minimal parking garage structure in \mathbb{R}^3 with two oppositely handed columns. We will treat each of these three cases separately.

After choosing a subsequence, suppose that the local picture of E for the sequence p_n is a catenoid. Since in the blow-up process we rescale the ambient metric, then after choosing an orthonormal basis for the metric Lie algebra of N , we can consider the ambient tangent directions as fixed and so, it makes sense to consider for n large, points $q_n \in E$ arbitrarily close to p_n so that the unit vectors $\nu_{q_n} \in T_{q_n} N$ converge as $n \rightarrow \infty$ to one of the two limiting normal vectors to the parallel ends of the limit catenoid. After choosing a subsequence, we can assume that the left translated vectors $G(q_n) = (l_{q_n}^{-1})_* \nu_{q_n} \in \mathbb{S}^2 \subset T_e N$ converge as $n \rightarrow \infty$ to a vector $w \in \mathbb{S}^2$.

We next briefly describe the set of two-dimensional subgroups of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ as we will use them in the argument; for details on the following description, see Section 2.7 in [4]. Recall that the projective special linear group $\mathrm{PSL}(2, \mathbb{R})$ can be considered to be the group $\mathrm{Iso}^+(\mathbb{H}^2)$ of orientation-preserving isometries of the hyperbolic plane \mathbb{H}^2 . Using the Poincaré disk model for \mathbb{H}^2 , for any point $\theta \in \mathbb{S}^1 = \partial_\infty \mathbb{H}^2$ one may consider the subset \mathbb{H}_θ of $\mathrm{PSL}(2, \mathbb{R})$ given by

$$\mathbb{H}_\theta = \{\phi \in \mathrm{Iso}^+(\mathbb{H}^2) \mid \phi(\theta) = \theta\}.$$

\mathbb{H}_θ is a connected two-dimensional subgroup of $\mathrm{PSL}(2, \mathbb{R})$, generated by the hyperbolic translations along geodesics of \mathbb{H}^2 one of whose ends points is θ and the parabolic translations along horocycles tangent at θ . Consider the covering map $\Pi: N = \widetilde{\mathrm{PSL}}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(2, \mathbb{R})$, which is a group homomorphism.

Let $\widetilde{\mathbb{H}}_\theta \subset \Pi^{-1}(\mathbb{H}_\theta)$ be the connected, two-dimensional subgroup of N passing through the identity. Since by definition, the left invariant Gauss map of a surface is invariant under left translations, it is clear that the left invariant Gauss map of a two-dimensional subgroup is constant. Particularizing to the case of the circle family of subgroups $\{\mathbb{H}_\theta \mid \theta \in \mathbb{S}^1\}$ of N , each $\widetilde{\mathbb{H}}_\theta$ has constant left invariant Gauss map $\Gamma(\theta) \in \mathbb{S}^2 \subset T_e \widetilde{\mathrm{PSL}}(2, \mathbb{R})$. Note that $\theta \in \mathbb{S}^1 \mapsto \Gamma(\theta) \in \mathbb{S}^2$ is injective, because the Gauss map image of a two-dimensional subgroup determines the subgroup itself.

Choose one of these two-dimensional subgroups $\widetilde{\mathbb{H}} = \widetilde{\mathbb{H}}_{\theta_0}$ so that the normal vector $\Gamma(\theta_0)$ is different from $\pm w$. As the ambient metric of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ is left invariant, it follows that the family of left cosets of $\widetilde{\mathbb{H}}$ forms a codimension-one foliation of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, all whose leaves are ambiently isometric to $\widetilde{\mathbb{H}}$ and have the same constant value $\Gamma(\theta_0)$ for their left invariant Gauss map as $\widetilde{\mathbb{H}}$. After choosing a subsequence of p_n , Theorem 1.5 implies that we can find a sequence $\varepsilon_n > 0$ converging to zero such that for each $n \in \mathbb{N}$, the closure C_n of the component of $E \cap B_N(p_n, \varepsilon_n)$ that contains p_n is a compact annulus and if n is large enough, C_n is arbitrarily close to large compact piece $C_\infty(n)$ of a complete catenoid $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$ and $I_{M_\infty}(\vec{0}) = 1$. In particular, $\vec{0}$ lies in the waist circle of M_∞ . Without loss of generality, we can assume that $C_\infty(n)$ is obtained by intersecting M_∞ with a ball of radius $n = \lambda_n \varepsilon_n$ in \mathbb{R}^3 centered at $\vec{0}$ (here $\lambda_n = 1/I_E(p_n)$, see items 2 and 3 of Theorem 1.5). Take $r > 0$ so that the waist circle γ_∞ of M_∞ lies in the ball of radius $r/2$ centered at $\vec{0}$. For n large, there exists a simple closed geodesic $\gamma_n \subset C_n$ such that $\lambda_n \gamma_n$ converges as $n \rightarrow \infty$ to γ_∞ . In particular for n large, γ_n lies in the ball $B_N(p_n, \delta_n)$, where $\delta_n = r/\lambda_n$ (hence $0 < \delta_n < \varepsilon_n$). Given $n \in \mathbb{N}$, let

$$U_n = \{q \in \widetilde{\mathrm{PSL}}(2, \mathbb{R}) \mid \mathrm{dist}(q, p_n \widetilde{\mathbb{H}}) < \delta_n\}$$

be the open tubular neighborhood of the topological plane $p_n \widetilde{\mathbb{H}}$ of radius δ_n , where dist denotes extrinsic distance in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$. Therefore the following property holds for all n large:

(P1) γ_n lies inside $B_N(p, \delta_n) \subset U_n$.

By Lemma 3.9 in [4], given a two-dimensional subgroup H of $\widetilde{\text{PSL}}(2, \mathbb{R})$, the family $\mathcal{F}(H) = \{Hx \mid x \in N\}$ of right cosets of H forms a codimension-one foliation of $\widetilde{\text{PSL}}(2, \mathbb{R})$ whose leaves have the property that each one is at a constant distance from each other. If we particularize to $H = H_n := p_n \widetilde{\mathbb{H}} p_n^{-1}$, it follows that each of the leaves of $\mathcal{F}(H_n)$ is at a constant distance from $H_n p_n = p_n \widetilde{\mathbb{H}}$. This implies that U_n is the union of the right cosets of H_n at distance less than δ_n from $p_n \widetilde{\mathbb{H}}$. Another consequence of this description is that $N - U_n$ is the set of points in N at distance at least δ_n from $p_n \widetilde{\mathbb{H}}$. Therefore, if we choose n large enough, the following property holds:

(P2) Each of the two boundary curves of C_n intersects the two components of $N - U_n$.

Let $E_n \subset E$ be the subannulus with boundary $\gamma_1 \cup \gamma_n$. Choose an integer $k \in \mathbb{N}$ sufficiently large, so that $\delta_k < \delta_1$. Since the two boundary components of U_1 are each right cosets of H_1 of constant distance $2\delta_1$ from each other, then the triangle inequality and property (P1) imply that the boundary curve γ_k of E_k does not intersect both of the boundary components ∂_1, ∂_2 of U_1 ; as γ_1 lies in between ∂_1, ∂_2 , then we deduce that γ_k is at the same side of at least one of the two components ∂_1, ∂_2 as γ_1 , say this component is ∂_1 . Property (P2) applied to C_1 gives that E_k contains points in the component Δ of $N - \partial_1$ that is disjoint from the boundary of E_k . By the compactness of E_k , there exists a point $q_k \in \Delta \cap E_k$ furthest from ∂_1 . By the above description of the leaves of $\mathcal{F}(H_1)$ as level sets of the distance function to ∂_1 , we deduce that the right coset $H_1 q_k \subset \Delta$ lies on one side of E_k at q_k . Since $\widetilde{\text{PSL}}(2, \mathbb{R})$ is a unimodular Lie group, then Corollary 3.17 in [4] implies that $H_1 q_k$ is a minimal surface, which gives a contradiction by the maximum principle for minimal surfaces applied at the point q_k , see Figure 1. This completes the proof of Case B when the local picture associated to the sequence p_n is a catenoid.

The proof of Case B when the associated local picture is a Riemann minimal example or a parking garage structure on \mathbb{R}^3 is essentially identical to the case of the local picture being a catenoid; in these two subcases, after choosing a subsequence, there are associated limit normal vectors $\pm w$ for the related ends of the Riemann minimal example or the planes of the foliation in the limit parking garage structure, just as was the case when the local picture was a catenoid. Once this choice of subsequence is made, the proof follows the same reasoning as in the catenoid case to obtain a contradiction, where in the case of a parking garage structure one uses the curves γ_k given in item 5.2 of Theorem 1.5. This completes the proof of the assertion when Case B holds.

Proof of Case C: Suppose N is a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with a left invariant metric, for some real 2×2 matrix A . As in the previous Case B, we will give details when the local picture associated to the sequence of points p_n of almost-minimal injectivity radius is a catenoid; the cases where this local picture is a Riemann minimal example or a parking garage structure can be argued similarly and we leave the details to the reader.

Assume that the local picture associated to the blow-up points p_n is a catenoid. Without loss of generality, we may assume that the constant mean curvature of the horizontal planes

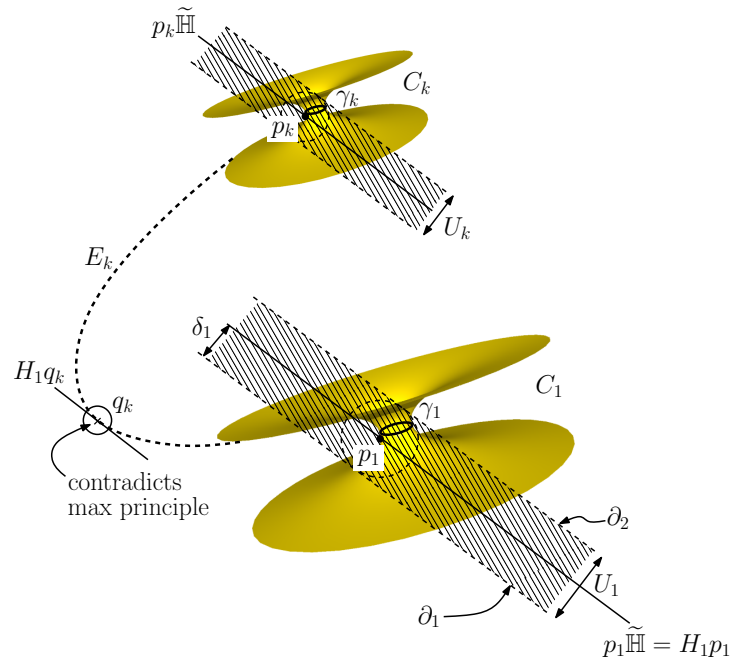


Figure 1: The boundary (topological) planes ∂_1, ∂_2 of U_1 are right cosets of H_1 at constant distance δ_1 from $H_1 p_1$. The contradiction comes from application of the maximum principle to the minimal surfaces $H_1 q_k$ and E_k at an appropriately chosen interior point q_k of E_k .

$x_3^{-1}(t) = \mathbb{R}^2 \rtimes_A \{t\}$ in N is $\text{trace}(A)/2 \geq 0$ with respect to their unit normal vector field $E_3 = \frac{\partial}{\partial x_3}$, which is left invariant (see Section 2.3 of [4]). By Remark 2.10 in [4], the level sets of the coordinate functions x_1, x_2 are minimal planes in N . It follows from the mean curvature comparison principle that if the x_3 -coordinate function of E has a local minimum at a point $q \in \text{Int}(E)$, then $E \subset \mathbb{R}^2 \rtimes_A \{x_3(q)\}$. Similarly, if the x_i -coordinate function, $i = 1, 2$, of E has a local minimum or maximum at a point $q \in \text{Int}(E)$, then $E \subset x_i^{-1}(x_i(q))$.

As in the proof of Case B, we may assume that after choosing a subsequence, the left invariant Gauss map of the ends of the limit local picture catenoid of E is $\pm w \in \mathbb{S}^2$. First consider the case where $w \neq \pm E_3(e)$. After choosing a subsequence of p_n , we can find a sequence $\varepsilon_n > 0$ converging to zero such that the compact catenoidal pieces $C_n \subset E \cap B_N(p_n, \varepsilon_n)$, which converge after blowing-up on the scale of topology to a large compact piece of a catenoid in \mathbb{R}^3 containing its waist circle, satisfy the following properties:

- (P1)' The almost-waist circle (simple closed geodesic) γ_n of C_n lies inside the open tubular neighborhood $U_n = \mathbb{R}^2 \rtimes_A \{x_3 \in \mathbb{R} \mid |x_3 - x_3(p_n)| < \delta_n\}$ of radius $\delta_n = r/\lambda_n$ of the horizontal plane $\mathbb{R}^2 \rtimes_A \{x_3(p_n)\}$ and inside $B_N(p_n, \delta_n)$ (here $\lambda_n = 1/I_E(p_n)$ and $r > 0$ are chosen as in the proof of Case B).
- (P2)' Each of the two boundary curves of C_n intersects the two components of $N - U_n$.

Given $n \in \mathbb{N}$, let $E_n \subset E$ be the subannulus with boundary $\gamma_1 \cup \gamma_n$. Then arguing as in the proof of Case B, one finds that for k large, the x_3 -coordinate function of E_k has its minimum value at an interior point q_k of E_k , which forces E to be an end representative of the plane $\{x_3 = x_3(q_k)\}$. Since this plane is isometric to the flat \mathbb{R}^2 but the injectivity radius function of E was assumed to take values arbitrarily close to zero away from its boundary, we obtain a contradiction. Henceforth, we will assume that $w = \pm E_3(e)$.

Since the local picture of E associated to the blow-up points p_n is a catenoid with a vertical axis, then it is clear that the x_1 -coordinate of a similarly defined subannulus E_k cannot have its global maximum at the boundary of E_k . Therefore, E must be contained in a vertical plane of the form $\{(t_0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$ for some $t_0 \in \mathbb{R}$. Exchanging x_1 by x_2 we find that E is contained in a plane of the form $\{(x_1, t'_0, x_3) \mid x_1, x_3 \in \mathbb{R}\}$ for some $t'_0 \in \mathbb{R}$, which is a contradiction. This completes the proof of Assertion 2.1. \square

Assertion 2.2 *Property (End) holds.*

Proof. Arguing by contradiction, assume that $p_n \in E$ is a divergent sequence in E with $I_E(p_n) < \frac{1}{n}$. As said in the second paragraph of the proof of Assertion 2.1, the points p_n can be assumed to be points of almost-minimal injectivity radius. Let $\lambda_n = 1/I_E(p_n)$. Since the scaled surfaces $\lambda_n M_n$ are minimal in $\lambda_n N$ and the sectional curvatures of the ambient spaces $\lambda_n N$ are converging uniformly to zero, then the Gauss equation implies that the exponential map \exp_{p_n} of $T_{p_n}(\lambda_n M_n)$ restricted to the closed metric ball of radius 2 centered at the origin

is a local diffeomorphism for n sufficiently large. As the injectivity radius of $\lambda_n M_n$ at p_n is 1, then \exp_{p_n} is a diffeomorphism when restricted to the open disk of radius 1, and it fails to be injective on the boundary circle of radius 1. Now it is standard to deduce the existence of simple closed geodesic loops Γ_n in E based at p_n , each of length less than $\frac{2}{n}$.

Let $\Pi: \tilde{N} \rightarrow N$ be the universal cover of N . We discuss two possibilities, depending on whether or not the geodesic loop Γ_n is homotopically trivial in E (note that after extracting a subsequence, we can assume all the Γ_n satisfy exactly one of the possibilities below).

(Q1) If all the Γ_n are homotopically nontrivial in E , then for n large f lifts through Π to a complete embedding $\tilde{f}: E \rightarrow \tilde{N}$, once we have picked an initial point in $\Pi^{-1}(f(\Gamma_n))$ (because as the lengths of the curves $f(\Gamma_n)$ tend to zero as $n \rightarrow \infty$, then for n large the image loops $f(\Gamma_n)$ lie in extrinsic balls in N of arbitrarily small radius and these extrinsic balls are topological balls for n sufficiently large because N has positive injectivity radius; hence, the induced map by f on the fundamental groups is trivial). In this setting, Assertion 2.1 gives a contradiction.

(Q2) All the Γ_n bound disks D_n in E . In this case, we consider a connected component \tilde{E} of $\Pi^{-1}(E)$ in \tilde{N} . Observe that the disks D_n lift through Π to a sequence of related disks $\tilde{D}_n \subset \tilde{E}$. It follows from the arguments in the proof of Theorem 1.5 applied to \tilde{E} at the sequence of preimages $\tilde{p}_n \in \partial \tilde{D}_n$ of the p_n (the \tilde{p}_n are points of almost-minimal injectivity radius for \tilde{E}) that the related local picture on the scale of topology of \tilde{E} around the \tilde{p}_n is either a catenoid, a Riemann minimal example or a minimal parking garage structure with two oppositely handed columns. As observed at the beginning of the proof of Assertion 2.1, this implies that the scalar flux $\text{Flux}(K_n^T, \partial \tilde{D}_n)$ with respect to certain Killing fields K_n in \tilde{N} , defined as in (1), is nonzero. As these fluxes are homological invariants associated to $\partial \tilde{D}_n$ but each $\partial \tilde{D}_n$ is homologically trivial on \tilde{E} , we obtain a contradiction. \square

As explained at the beginning of the proof of Theorem 1.1, the validity of Property (End) finishes the proof of the theorem. \square

Proof of Corollary 1.2. Let M be a complete embedded minimal surface of finite topology in a complete, locally homogeneous three-manifold N . In order to conclude that \overline{M} has the structure of a minimal lamination, it suffices to check that the injectivity radius function of M is bounded away from zero in every extrinsic ball of N , see Remark 2 in [10]. This extrinsic local positivity of the injectivity radius function of M follows directly from modifications of the lifting arguments in the proof of Assertion 2.2; the only modification occurs in case (Q1) above, where we must replace the hypothesis that the injectivity radius of N is positive by the property that the injectivity radius function I_N in any compact extrinsic ball of N is bounded away from zero. Therefore, \overline{M} is a minimal lamination of N .

By the Stable Limit Leaf Theorem for H -laminations in [6, 7], the limit leaves of \overline{M} have the property that their two-sided covers are stable. This proves item 1 of the corollary.

Assume now that N has positive scalar curvature (as N is locally homogeneous, then its scalar curvature is constant). If M is not proper in N , then \overline{M} contains a limit leaf L . Since the two-sided cover of L is stable, it follows from item 1 of Theorem 2.13 in [6] that L is either an embedded sphere or an embedded projective plane. After lifting the lamination \overline{M} to a two-sheeted cover of N , we may assume that L is a sphere. But this is impossible since it is a leaf of a lamination and so the nearby leaves must also be spheres which is not true since M is noncompact. This contradiction proves item 2 in the corollary.

To prove item 3 of Corollary 1.2, assume that N has nonnegative scalar curvature and it is simply connected. By item 2 of the corollary, we may assume that the scalar curvature of N vanishes. Since N is homogeneous, then, metric classification results in Section 4 of Milnor [11] imply that N is either isometric to \mathbb{R}^3 or N is $SU(2)$ with a left invariant metric. By the main result of Colding and Minicozzi in [2], if N is flat, then M is proper (in fact, M has positive injectivity radius by the results in [10]).

To finish the proof of item 3 of the corollary, it remains to demonstrate that if N is $SU(2)$ endowed with a left invariant metric of zero scalar curvature, then M is proper. We will show that M is indeed compact, even in the more general case that the constant scalar curvature of the left invariant metric on $SU(2)$ is nonnegative, thereby proving also item 4 of Corollary 1.2 (this will then finish the proof of the corollary).

Suppose that M is noncompact and we will derive a contradiction. As M is noncompact, then the minimal lamination \overline{M} has a limit leaf L whose two-sided cover \tilde{L} must be stable. Since the scalar curvature of N is assumed to be nonnegative, then item 2(a) of Theorem 2.13 in [6] implies that \tilde{L} has at most quadratic area growth. In this situation, it follows from Theorem 1 in [3] that the space of bounded Jacobi functions on \tilde{L} is one-dimensional, and this space coincides with the space of Jacobi functions with constant sign on \tilde{L} . Consider the three-dimensional space of right invariant Killing fields on N . As N is compact, then every such vector field F is bounded on N , and thus, the inner product $\langle F, \nu \rangle$ of F with the unit normal vector field ν to \tilde{L} defines a bounded Jacobi function on \tilde{L} . Now pick a point $p \in \tilde{L}$ and let F_1, F_2 be linearly independent right invariant vector fields on $SU(2)$ so that $F_1(p), F_2(p)$ are tangent to \tilde{L} at p . Since for $i = 1, 2$ the function $u_i = \langle F_i, \nu \rangle$ is a bounded Jacobi function on \tilde{L} , then u_i has constant sign on \tilde{L} . As u_i vanishes at p , then F_i is everywhere tangent to \tilde{L} , for $i = 1, 2$. In particular, the Lie bracket $[F_1, F_2]$ is also everywhere tangent to \tilde{L} . This is impossible, as $[F_1, F_2]$ is a right invariant vector field on $SU(2)$ which is everywhere linearly independent with F_1, F_2 (this last property follows from the structure constants of the unimodular group $SU(2)$). This contradiction implies that M must be compact, which completes the proof of Corollary 1.2. \square

Remark 2.3 We now prove the result stated in the last sentence of the abstract. To see this property holds first observe that, by the arguments at the end of the proof of Corollary 1.2, $N = \mathbb{S}^3$ equipped with a homogeneous metric with nonnegative scalar curvature satisfies the hypothesis of N described in the next corollary, which means that any complete embedded

minimal surface in N cannot have an annular end. Next observe that if Σ is a surface of finite genus with countably many ends, then Σ is compact or it must have at least one annular end, and so, the stated property in the abstract now follows.

Corollary 2.4 *Let N be a complete, locally homogeneous three-manifold which does not admit any complete embedded stable minimal surfaces. If M is a complete embedded minimal surface in N , then every annular end of M is proper.*

Proof. Suppose $E \subset M$ is an annular end. With minor modifications, the proof of Corollary 1.2 shows that the injectivity radius function of M restricted to E is bounded away from zero on compact subdomains of N . Suppose that E is not proper. Then, the limit set $\text{Lim}(E)$ of E is nonempty. As the injectivity radius function of M restricted to E is bounded away from zero on compact subdomains of N , the Lamination Closure Theorem in [10] implies that $\text{Lim}(E)$ has the structure of a minimal lamination of N . By the Stable Limit Leaf Theorem for H -laminations [6, 7], the two-sided cover of any leaf of $\text{Lim}(E)$ is stable, thereby contradicting our hypotheses on N . \square

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